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Noisy One-Dimensional Maps Near a Crisis. II. General Uncorrelated Weak Noise

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The escape rate for one-dimensional noisy maps near a crisis is investigated. A previously introduced perturbation theory is extended to very general kinds of weak uncorrelated noise, including multiplicative white noise as a special case. For single-humped maps near the boundary crisis at fully developed chaos an asymptotically exact scaling law for the rate is derived. It predicts that transient chaos is stabilized by basically any noise of appropriate strength provided the maximum of the map is of sufficiently large order. A simple heuristic explanation of this effect is given. The escape rate is discussed in detail for noise distributions of Lévy, dichotomous, and exponential type. In the latter case, the rate is dominated by an exponentially leading Arrhenius factor in the deep precritical regime. However, the preexponential factor may still depend more strongly than any power law on the noise strength.

KEY WORDS: Noisy map; crisis; escape rate; structural instability; dichotomous noise; Lévy distribution.

1. INTRODUCTION AND SUMMARY

Of central interest in the study of nonlinear dynamical systems is the characterization of their time evolution as a function of some control parameters (see ref. 1 for review). Parameter values at which a chaotic attractor collides with a coexisting unstable fixed point or periodic orbit are called *crises*.[2] The sudden qualitative changes of the time evolution that occur at a crisis have received considerable attention during the recent years and it was soon realized that in many cases the influence of noise on these highly sensitive phenomena plays an important role.[3-17] Closely

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related topics have also been investigated under the labels "noise-induced chaos,"[18] "generalized multistability,"[19] "noise-induced transitions in chaotic systems,"[20] certain type of deterministic diffusion with noise,[21,22] and the decay of metastable states with fractal basin boundaries.[23,24]

In this article we continue our study[25] (henceforth referred to as I) of one-dimensional maps near crises in the presence of weak random fluctuations. In Section 2 our model system is introduced, consisting of a one-dimensional Markovian process in discrete time. It describes the dynamics of a particle x under the simultaneous action of a single-humped map $f(x)$ and a noise ξ of small intensity σ. A second small parameter Δ measures the distance frow a boundary crisis at $\Delta = 0$. In the absence of noise ($\sigma = 0$) the deterministic dynamics at the crisis $\Delta = 0$ exhibits fully developed chaos[26,27] on the unit interval $[0, 1]$ and the corresponding invariant density $\rho(x)$ is assumed to be known. Besides the smallness of Δ and σ, the conditions regarding the map $f(x)$ and the noise ξ are very weak. In particular, the maximum of the map at $x = x^*$ may be of arbitrary order $z > 0$ and the noise distribution $P(\xi, x)$ may explicitly depend on the present state x of the system in a very general way.

In Section 3 we generalize the perturbation-like method introduced in I in order to determine the probability distribution of particles in the quasistationary state for small values of Δ and σ. This method is valid under the additional necessary and sufficient condition (23), notwithstanding the periodic windows[1] of the map $f(x)$ that may occur for arbitrarily small $\Delta < 0$ when $z > 1$. Put differently, under this extra condition on σ and Δ we are able to overcome the problems arising from the fact that the deterministic dynamics is *structurally unstable* at the crisis.[28] These possible complications have always been tacitly ignored in similar investigations[3,4,8,11,13,15] dealing with the effects of weak noise in the precritical regime ($\Delta < 0$), for instance, so-called noise-induced crises.

From the probability distribution in the quasistationary state we then obtain our central result (39) for the escape rate k of particles out of the unit interval $[0, 1]$. This rate formula has the form of a *scaling law*[4,11] and becomes asymptotically exact for small Δ and σ satisfying the extra condition (23). Unlike in previously derived approximations,[3,4,9,11] possible recrossings of the interval boundaries 0 and 1 by the particles are fully taken into account.

In Section 4 the rate formula is discussed in detail for a variety of multiplicative white noises. Specifically, if the noise is governed by a Lévy distribution [see Eq. (11)] one can identify universality classes, characterized by common critical exponents *and* scaling functions in the scaling law for the rate. For exponential noise distributions [see Eq. (15)] the recently predicted[15] exponentially dominating Arrhenius factor in the deep precritical regime [$\Delta/\sigma \ll -1$ but still respecting (23)] is determined explicitly. Further, it is demonstrated that the preexponential "correction" of this dominating contribution to the rate may actually be very strong. For a particular kind of multiplicative noise acting on the logistic map, a comparison with numerical simulations is given in Fig. 1.

In Section 5 we focus on the dependence of the escape rate upon the noise strength σ in the postcritical regime ($\Delta > 0$), corresponding to transient chaos (see ref. 29 for review) in the absence of noise. We show that the rate for $\sigma = 0$ is reduced by basically any kind of noise, provided the maximum of the map is of sufficiently large order z and σ is chosen appropriately. In particular, for symmetric noise distributions $P(-\xi, x) = P(\xi, x)$ it is sufficient (and necessary) that $z > 1$. This phenomenon that deterministic transient chaos may be stabilized by weak noise was first observed in a numerical study by Franaszek.[10] Here, we analytically demonstrate the universality of this effect and we also offer a simple intuitive explanation.

2. NOISY MAPS NEAR BOUNDARY CRISES

We consider the one-dimensional dynamics of a particle with coordinate x in discrete time n,

$$x_{n+1} = f(x_n) + \sigma \xi_n \tag{1}$$

where $f(x)$ is a map of the real axis, ξ_n represents the noise, and the noise strength σ is small,

$$0 \leqslant \sigma \ll 1 \tag{2}$$

The map $f(x)$ is assumed to be single-humped with a maximum of order $z > 0$ at x^* and to be given in leading order close to this maximum by

$$f(x + x^*) = 1 + \Delta - b\,|x|^z \tag{3}$$

where $b > 0$ and Δ is a second small parameter,[3]

$$-1 \ll \Delta \ll 1 \tag{4}$$

We further assume that $f(x)$ is continuously differentiable with $f'(x) \neq 0$ for all $x \neq x^*$. The x scale is chosen such that $x = 0$ is an unstable fixed point

[3] Strictly speaking, we consider families of maps $f(x)$ that are parametrized by Δ. However, if this parametrization is smooth, then for sufficiently small Δ the dependence of $f(x)$ on Δ turns out to be negligible except for Δ itself in (3).

and a second zero of $f(x)$ is at $x=1$. This implies the following relations: $f(0)=0$, $f'(0)>1$, $0<x^*<1$, $f(1)=0$, and $f'(1)<0$. A well-known example is the logistic map

$$f(x)=4(1+\Delta)\,x(1-x) \tag{5}$$

where obviously $z=2$, $x^*=1/2$, and $b=f'(0)=-f'(1)=4(1+\Delta)$.

For $\Delta=0$ the unit interval $[0, 1]$ is mapped onto itself by $f(x)$ and there exists a unique *invariant density* $\rho(x)$ describing the stationary state of the dynamics (1) in the deterministic limit $\sigma=0$. For instance, in the case of the logistic map (1) one has[1]

$$\rho(x)=\frac{1}{\pi}\frac{1}{\sqrt{x(1-x)}} \tag{6}$$

if $x\in[0, 1]$ and $\rho(x)=0$ otherwise. In the following we restrict ourselves to maps $f_{\Delta=0}(x)$ with an invariant density $\rho(x)$ that is positive, bounded, and continuous on the unit interval with the exception of arbitrarily small neighborhoods of the boundaries $x=0$ and $x=1$. (We conjectured in I that this will be guaranteed under the necessary and sufficient conditions that $f_{\Delta=0}(x)$ has no stable fixed points or stable periodic orbits on $[0, 1]$ or that $\rho(x)\neq 0$ close to x^*.) It follows that the map $f(x)$ shows fully developed chaos[26,27] and exhibits a boundary crisis[2] when $\Delta=0$. For $\Delta>0$ we are dealing with transient chaos[29] and for $\Delta<0$ with permanent chaos or with a periodic window[1] (the latter possibility can be ruled out for $z\leqslant 1$ and sufficiently small $\Delta<0$).

The noisy dynamics (1) is assumed to be an autonomous Markovian process. In other words, the noise ξ_n is governed by a probability distribution $P(\xi, x)$ which is allowed to depend explicitly on the present state $x=x_n$ of the system but not on the time step n nor on the state x_m, and the noise ξ_m at times $m\neq n$:

$$\mathrm{Prob}(\xi_n\in[\xi, \xi+d\xi])=P(\xi, x=x_n)\,d\xi \tag{7}$$

Thus, the normalization condition reads $\int_{-\infty}^{\infty} P(\xi, x)\,d\xi=1$ for any x. We further require that there exist x-independent positive numbers η and ξ_0 (they may be arbitrarily small and large, respectively) such that

$$P(\xi, x)\leqslant 1/|\xi|^{1+\eta+1/z} \tag{8}$$

whenever $|\xi|\geqslant\xi_0$. [Through z this is actually a joint condition regarding both the map $f(x)$ and the noise distribution.] Finally, we assume that

$P(\xi, x)$ can be approximated by $P(\xi, x^*)$ for all x sufficiently close to x^* and similarly for x close to 0 or 1, i.e.,

$$P(\xi, x) \to P(\xi, \hat{x}) \qquad \text{for} \quad x \to \hat{x} \tag{9}$$

where $\hat{x}$ represents any of the points $0, x^*$, or 1. In full generality, this condition (8) *has to be understood in the sense of distributions*, i.e., the integral $\int_{-\infty}^{\infty} t(\xi)\, P(\xi, x)\, d\xi$ is supposed to be finite for any test function $t(\xi)$ that is continuous and increases at most like $|\xi|^{1/z}$ for large positive and negative ξ. Similarly, (9) means that $\int_{-\infty}^{\infty} t(\xi)[P(\xi, x) - P(\xi, \hat{x})]\, d\xi \to 0$ for $x \to \hat{x}$ and any such test function $t(\xi)$. For instance, $P(\xi, x)$ may exhibit δ-peaks at arbitrarily large ξ but with sufficiently small weights. Moreover, the positions and weights of such δ-peaks must vary continuously with x in the neighborhoods of $0, x^*$, and 1.

The dynamics (1) with a noise distribution of the form

$$P(\xi, x) = P_M(\xi/g(x))/|g(x)| \tag{10}$$

is equivalent to the dynamics $x_{n+1} = f(x_n) + \sigma g(x_n)\, \xi_n$ with a noise distribution $P_M(\xi)$ that is independent of the state x of the system. The conditions (8) and (9) are satisfied if $P_M(\xi)$ fulfills (8) and $g(x)$ is bounded on $\mathbf{R}$ and continuous at $0, x^*$ and 1. Within this restriction, *multiplicative noise* is thus a special case (10) in the general class of noise distributions (7) considered here.

We close this section with three examples of multiplicative noise distributions (10) that fulfill the condition (8) and will be considered in more detail in Section 4.

1. The symmetric Lévy distributions[30]

$$P_M(\xi) = \int_{-\infty}^{\infty} \frac{dq}{2\pi}\, e^{iq\xi} e^{-|q|^\mu}, \qquad 0 < \mu \leqslant 2 \tag{11}$$

In particular, one recovers a Gaussian distribution for $\mu = 2$,

$$P_M(\xi) = (4\pi)^{-1/2}\, e^{-\xi^2/4} \tag{12}$$

and a Lorentz distribution for $\mu = 1$,

$$P_M(\xi) = \frac{1}{\pi} \frac{1}{1 + \xi^2} \tag{13}$$

Note that for $\mu < 2$ the asymptotic behavior for large ξ is $P_M(\xi) \sim 1/|\xi|^{1+\mu}$.[30] Hence, for symmetric Lévy distributions with $\mu < 2$ we will tacitly restrict ourselves to maps $f(x)$ with maxima of order $z > 1/\mu$ in view of the condition (8).

2. The "dichotomous white noise"[31,32]

$$P_M(\xi) = \delta(|\xi| - 1)/2 \tag{14}$$

3. The symmetric "exponential distributions"[15,33,34]

$$P_M(\xi) = \frac{\alpha}{2\Gamma(1/\alpha)} \exp\{-|\xi|^\alpha\}, \qquad \alpha > 0 \tag{15}$$

The specific α values 1, 2, and ∞ correspond to ordinary exponential, Gaussian, and confined homogeneous[10,11,35,36] noise distributions, respectively. It should be emphasized that unlike in the examples (11)–(15), we will in general *not* assume symmetry of the noise distributions (7) about $\xi = 0$.

3. THE ESCAPE RATE

3.1. General Framework

The dynamical system (1)–(4) specified in the preceding section approaches a quasistationary state for large times n.[3,4] This means that the probability distribution of the particles described by (1) becomes proportional to a *quasi-invariant density* $W(x)$, where the proportionality constant differs very little for two successive time steps n and $W(x)$ is n independent. It follows that the quasi-invariant density fulfills in very good approximation the master equation

$$W(x) = \int_{-\infty}^{\infty} P(x \mid y)\, W(y)\, dy \tag{16}$$

where the transition probability $P(x \mid y)$ that a particle jumps from y to x in one time step is given by

$$P(x \mid y) = \frac{1}{\sigma} P\left(\frac{x - f(y)}{\sigma}, y\right) \tag{17}$$

according to (1) and (7). Moreover, the rate k at which particles escape from the unit interval in the quasistationary state can be written as

$$k = \frac{\int_0^1 [W(x) - \int_{-\infty}^{\infty} P(x \mid y)\, W(y)\, dy]\, dx}{\int_0^1 W(x)\, dx} \tag{18}$$

As detailed in I, even if $W(x)$ is an approximate solution of the master equation (16) which is not exactly equal to the true quasi-invariant density, the escape rate (18) which one derives from this approximation will be equal to the true rate within the same accuracy. Such an approximate solution $W(x)$ and the resulting rate k will be determined in the following Sections 3.2 and 3.3, respectively. It can be shown by closer inspection that these approximations for the quasi-invariant density and the escape rate become asymptotically exact for small noise strengths σ and parameters Δ.

3.2. The Quasi-Invariant Density

Due to the required smoothness of the map $f(x)$ and the continuity of the noise distribution (9) it is possible to find two small positive quantities ε_i, $i = 1, 2$, such that in arbitrarily good approximation

$$f(x) = f'(1)(x-1), \qquad P(\xi, x) = P(\xi, 1) \qquad \text{for} \quad x \in [1-\varepsilon_1, 1+\varepsilon_1] \tag{19}$$

$$f(x) = f'(0)\, x, \qquad P(\xi, x) = P(\xi, 0) \qquad \text{for} \quad x \in [-\varepsilon_2, \varepsilon_2] \tag{20}$$

Since we are interested in asymptotically small noise strengths σ and parameters Δ, we can always assume in the following that ε_1 and ε_2 are much larger than σ and $|\Delta|$. It then is obvious that the probability for a particle which has left the interval $[-\varepsilon_2, 1+\varepsilon_1]$ to return into this interval is negligible. We thus can set

$$W(x) = 0 \qquad \text{for} \quad x \notin [-\varepsilon_2, 1+\varepsilon_1] \tag{21}$$

without notably changing $W(x)$ at those x values which mainly contribute to the rate (18).

At the crisis ($\Delta = 0$) and in the absence of noise ($\sigma = 0$) the quasi-invariant density $W(x)$ coincides with the invariant density $\rho(x)$. As discussed below Eq. (6), the invariant density $\rho(x)$ is positive, bounded, and continuous on $[\varepsilon_2, 1-\varepsilon_1]$, but typically has singularities at 0 and 1 and vanishes outside $[0, 1]$. This suggests that even for nonvanishing but sufficiently small σ and Δ the functions $W(x)$ and $\rho(x)$ will still approximately agree on $[\varepsilon_2, 1-\varepsilon_1]$,

$$W(x) = \rho(x) \qquad \text{for} \quad x \in [\varepsilon_2, 1-\varepsilon_1] \tag{22}$$

while outside this interval they may substantially differ. A complication arises from the periodic windows[1] occurring for maps $f(x)$ with maxima of order $z > 1$ in the regime $\Delta < 0$, since in such a case Eq. (22) becomes obviously wrong in the deterministic limit $\sigma \to 0$. Additionally, $W(x)$ typically develops singularities in the domain $[\varepsilon_2, 1-\varepsilon_1]$ when $z>1$ and $\sigma \to 0$ even for those $\Delta<0$ not corresponding to a periodic window (see, e.g., Fig. 3 in I). In other words, $W(x)$ is structurally unstable at the crisis in the deterministic limit.[28] By closer inspection of this difficulty the following necessary and sufficient condition for the validity of (22) was derived in I for the special case of additive Gaussian white noise:

$$\sigma \gg |\Delta|^{z/(z-1)} \qquad \text{for} \quad z>1, \quad \Delta<0 \tag{23}$$

Loosely speaking, it guarantees that the singularities of $W(x)$ on $[\varepsilon_2, 1-\varepsilon_1]$ that one would encounter in the limit $\sigma \to 0$ are sufficiently "washed out" by the noise. It can be shown that the line of reasoning in I to prove (23) can be extended to the entire class of noise distributions $P(\xi, x)$ considered here. Note that (23) only concerns $z>1$ and $\Delta<0$. For $z \leqslant 1$ or $\Delta \geqslant 0$ there is no additional condition in order that (22) is valid, since periodic windows are ruled out. From now on we always restrict ourselves to small noise strengths (2) and parameters (4) that satisfy the extra condition (23). we are thus left to determine the quasi-invariant density $W(x)$ in the regions $[1-\varepsilon_1, 1+\varepsilon_1]$ and $[-\varepsilon_2, \varepsilon_2]$.

In order to determine $W(x)$ for $x \in [1-\varepsilon_1, 1+\varepsilon_1]$ one exploits the fact suggested by (17) and (8) and discussed in more detail in I that only y values close to the maximum x^* of $f(x)$ notably contribute to the integral in the master equation (16). For these y values one can use the approximations (3) for $f(y)$ and $P(\xi, x) = P(\xi, x^*)$ according to (9). Further, due to (22) and the continuity of $\rho(x)$ at $x = x^*$ one may approximate $W(y)$ by $\rho(x^*)$. Finally, all these approximations can be extended to arbitrary y values without notably changing the value of the integral in the master equation (16). We thus find that for $x \in [1-\varepsilon_1, 1+\varepsilon_1]$

$$W(x) = \rho(x^*) \int_{-\infty}^{\infty} \frac{dy}{\sigma} P\left(\frac{x-1-\Delta+b\,|y|^z}{\sigma}, x^*\right) =: w_0(x-1) \tag{24}$$

Similarly, in order to determine $W(x)$ for $x \in [-\varepsilon_2, \varepsilon_2]$ one observes that only values of y belonging to the interval $[-\varepsilon_2, \varepsilon_2]$ or a small neighborhood of 1 give nonnegligible contributions to the integral in the master equation (16) (for details see I). Without loss of generality, ε_1 and ε_2 can be chosen such that the latter neighborhood of 1 coincides with $[1-\varepsilon_1, 1+\varepsilon_1]$. For $y \in [-\varepsilon_2, \varepsilon_2]$ one then can use (19) and for $y \in [1-\varepsilon_1, 1+\varepsilon_1]$, (20) and (24). Finally, one again can extend these approximations to arbitrary y values without notably changing the value of the integral in the master equation (16). Hence, for $x \in [-\varepsilon_2, \varepsilon_2]$ the master equation (16) is equivalent to

$$W(x) = \int_{-\infty}^{\infty} \frac{dy}{\sigma} P\left(\frac{x - f'(0)\, y}{\sigma}, 0\right) W(y) + \int_{-\infty}^{\infty} \frac{dy}{\sigma} P\left(\frac{x - f'(1)\, y}{\sigma}, 1\right) w_0(y) \tag{25}$$

Next we will determine a solution $W(x)$ of (25) on the whole real axis. However, it should be noted that this $W(x)$ represents the quasi-invariant density, i.e., an approximate solution of the master equation (16), only for $x \in [-\varepsilon_2, \varepsilon_2]$. Since (25) is an inhomogeneous Fredholm integral equation, we may expect a solution of the form

$$W(x) = \sum_{i=1}^{m} w_i(x) \tag{26}$$

$$w_i(x) = \int_{-\infty}^{\infty} \frac{dy}{\sigma} P\left(\frac{x - f'(x_i^*)\, y}{\sigma}, x_i^*\right) w_{i-1}(y), \qquad i \geqslant 1 \tag{27}$$

where m is kept finite for the moment, $w_0(x)$ is given in (24), and

$$x_i^* := f^i_{\Delta=0}(x^*), \qquad i \geqslant 0 \tag{28}$$

In other words, we have $x_0^* = x^*$, $x_1^* = 1$, and $x_i^* = 0$ for $i \geqslant 2$. By means of a straightforward calculation one finds that (24), (27) can be rewritten in the form

$$w_i(x) = \frac{2p(x^*)\, \sigma^{-1+1/z}}{z b^{1/z}\, |A_i|} \int_0^{\infty} dy\, y^{-1+1/z} h_i\left(y + \frac{x/A_i - \Delta}{\sigma}\right) \tag{29}$$

$$h_i(x) = \int_{-\infty}^{\infty} dy\, |A_i|\, P(A_i(x - y), x_i^*)\, h_{i-1}(y), \qquad i \geqslant 1 \tag{30}$$

$$h_0(x) = P(x, x^*) \tag{31}$$

where we introduced

$$A_i := \prod_{l=1}^{i} f'(x_l^*) = \frac{d}{dx} f^i_{\Delta=0}(x = x^*), \qquad i \geqslant 0 \tag{32}$$

Here, for $i = 0$ the product is defined to be 1 and in the last equality we used the approximation $f'(x_l^*) = f'_{\Delta=0}(x_l^*)$ valid for small Δ [see also the paragraph following Eq. (5)]. We thus have $A_0 = 1$ and $A_i = f'(1)\, f'(0)^{i-1}$ for $i \geqslant 1$. As mentioned below Eq. (4), $f'(1) < 0$ and $f'(0) > 1$, implying that the A_i are negative for $i \geqslant 1$ and diverge exponentially toward $-\infty$ for large i.

From (30), (31) it follows by induction that

$$\int_{-\infty}^{\infty} h_i(x)\,dx = 1 \tag{33}$$

Further, it can be concluded from (8) and (30) by induction that

$$h_i(x) \leqslant 1/|x|^{1+\eta+1/z} \qquad \text{for} \quad |x| \gg 1 \tag{34}$$

In particular, this implies that the integrals in (29) are finite and that the main contribution to the integral in (33) stems from a finite (σ- and Δ-independent) neighborhood of $x = 0$. Taking into account (29) and (34), a simple calculation yields the asymptotic upper bound

$$w_i(x) \leqslant \frac{2(1+1/z)\,\rho(x^*)\,\sigma^{\eta+1/z}\,|A_i|^{\eta}}{b^{1/z}\,|x - A_i\Delta|^{1+\eta}} \qquad \text{for} \quad \frac{x}{A_i} - \Delta \gg \sigma \tag{35}$$

where we recall that $A_0 = 1$ and $A_i < 0$ for $i \geqslant 1$. Next we address the asymptotic behavior of $w_i(x)$ for $x/A_i - \Delta \ll -\sigma$: In this case it follows from (33) and (34) that the main contribution to the integral (29) stems from a small neighborhood of $y = (-x/A_i + \Delta)/\sigma \gg 1$. Since in this neighborhood $y^{1/z-1}$ is practically constant and taking into account (33), it then follows that

$$w_i(x) = \frac{2\rho(x^*)\,|x - A_i\Delta|^{-1+1/z}}{z b^{1/z}\,|A_i|^{1/z}} \qquad \text{for} \quad \frac{x}{A_i} - \Delta \ll -\sigma \tag{36}$$

Remarkably, the asymptotics (36) is completely independent of the noise distribution $P(\xi, x)$. In the limit $\Delta = \sigma = 0$ one recovers from (24), (26), (35), and (36) the correct features of $\rho(x)$ outside $[\varepsilon_2, 1-\varepsilon_1]$ mentioned above Eq. (22).

For $z \leqslant 1$ a partial integration of (29) yields $w_i'(x) \geqslant 0$ for $i \geqslant 1$. Thus, $w_i(x)$ is monotonically increasing. Similarly, one can see that $w_0(x)$ is monotonically decreasing. On the other hand, for $z > 1$ and any $i \geqslant 0$ it can be shown from Eqs. (33)–(36) that $w_i(x)$ in (29) has an absolute maximum in the domain $|x/A_i - \Delta| \leqslant O(\sigma)$ and can be roughly approximated by

$$w_i(x) \simeq w_i(A_i\Delta) = \frac{2\rho(x^*)\int_0^{\infty} dy\, y^{-1+1/z} h_i(y)}{z b^{1/z}\,|A_i|\,\sigma^{1-1/z}} \qquad \text{for} \quad \left|\frac{x}{A_i} - \Delta\right| \leqslant O(\sigma) \tag{37}$$

[The approximation (37) is actually valid also for $z \leqslant 1$.] These findings together with the asymptotic properties (35) and (36) provide a good qualitative picture of the functions $w_i(x)$.

Since A_i tends to $-\infty$ for large i, the kernel $|A_i|\, P(A_i(x-y), x_i^*)$ in the integral (30) approaches $\delta(x-y)$. The fact that the divergence of A_i is exponential in i then suggests that this convergence toward $\delta(x-y)$ is sufficiently fast in order that the functions $h_i(x)$ also tend toward a well-defined limit $h_\infty(x)$. Note that in the most general case this convergence $h_i(x) \to h_\infty(x)$ is understood in the sense of distributions; see the discussion following Eq. (9). For the particular noise distributions (11)–(15), the existence of $h_\infty(x)$ will be explicitly verified in Section 4. However, from a rigorous point of view we cannot prove the existence of $h_\infty(x)$ in general[4] and it is thus henceforth understood as a tacit additional assumption regarding the noise distribution $P(\xi, x)$ [and strictly speaking also the values of $f'(0)$, $f'(1)$, and x^*]. With (37) it then follows that for large i the functions $w_i(x)$ become x-independent on the interval $[-\varepsilon_2, \varepsilon_2]$ and decrease proportional to $f'(0)^{-i}$. As a consequence, $W(x)$ from (26) and (29) tends toward an m-independent finite limit and solves (25) in arbitrarily good approximation for $x \in [-\varepsilon_2, \varepsilon_2]$ and sufficiently large m. Furthermore, it can be shown by exploiting the properties (35)–(37) of $w_i(x)$ and the existence of $h_\infty(x)$ that $W(x)$ from (26), (29) goes over into the correct behavior (21) and (22) for x values close to the boundaries of the intervals $[-\varepsilon_2, \varepsilon_2]$ and $[1-\varepsilon_1, 1+\varepsilon_1]$. In other words, we found a self-consistent approximate solution $W(x)$ of the master equation (16) on the whole real axis.

3.3. The Rate Formula

We first evaluate the denominator $\int_0^1 W(x)\, dx$ in the escape rate (18). According to (22), we have $\rho(x) = W(x)$ on the major part of the unit interval $[0, 1]$. It is suggestive and can be rigorously shown by means of the results from Section 3.2 that contributions of x values from the small neighborhoods of 0 and 1 where $W(x)$ and $\rho(x)$ notably differ, are negligible both in $\int_0^1 W(x)\, dx$ and $\int_0^1 \rho(x)\, dx$. Since $\rho(x)$ is normalized on the unit interval, the denominator $\int_0^1 W(x)\, dx$ in the escape rate (18) can thus be approximated by 1. The evaluation of the numerator is somewhat more involved. However, the basic step, namely the asymptotic equality

$$k = \lim_{m \to \infty} \int_{-\infty}^{0} w_m(x)\, dx \tag{38}$$

[4] A similar mathematical problem (but within a rather different physical context) has been studied in ref. 37. For a restricted class of noise distributions $P(\xi, x)$, the existence of $h_\infty(x)$ follows from Theorem 3.1 therein.

for small σ and Δ is obtained by a straightforward adaptation of the line of reasoning detailed in I.

Introducing (29) into (38), we can rewrite the rate in the form

$$k = \rho(x^*)(\sigma/b)^{1/z} F(\Delta/\sigma) \tag{39}$$

$$F(x) := 2\int_0^\infty dy\, y^{1/z} h_\infty(y-x) \tag{40}$$

This is the central result of our paper. It becomes asymptotically exact for small noise strengths σ and distances Δ from the crisis under the additional necessary and sufficient condition (23). In (39), $\rho(x)$ is the invariant density at the crisis in the absence of noise normalized on the unit interval. The quantities b and z describe the map $f(x)$ near its maximum x^*; see Eq. (3). The existence of the limit $h_\infty(x) = \lim_{i\to\infty} h_i(x)$ is implicitly assumed. It can be determined by means of the recursion (30), (31), which is particularly suitable for numerical purposes. On the other hand, a Fourier transformation of (30), (31) yields

$$\tilde{h}_i(q) = \int_{-\infty}^{\infty} h_i(x)\, e^{-iqx}\, dx = \prod_{l=0}^{i} \tilde{P}(q/\Lambda_l, x_l^*) \tag{41}$$

where the Fourier transform of the noise distribution (7) is given by

$$\tilde{P}(q, x) := \int_{-\infty}^{\infty} P(\xi, x)\, e^{-iq\xi}\, d\xi \tag{42}$$

By a Fourier backtransformation and using (28), (32), one thus arrives at

$$h_\infty(x) = \int_{-\infty}^{\infty} \frac{dq}{2\pi} e^{iqx} \prod_{l=0}^{\infty} \tilde{P}\left(\frac{q}{(d/dx)\, f^l_{\Delta=0}(x=x^*)}, f^l_{\Delta=0}(x^*)\right) \tag{43}$$

This expression for $h_\infty(x)$ gives the explicit connection between $F(x)$ in (40) and the noise distribution $P(\xi, x)$. It is particularly useful in cases which can be analytically solved; see Section 4. Note that the only properties of the map $f(x)$ entering $h_\infty(x)$ are the maximum x^*, its iterates 1 and 0, and the derivatives $f'(1)$ and $f'(0)$ at the specific parameter value $\Delta = 0$. In particular, the actual values of Δ, z, and b describing the behavior of $f(x)$ near x^* do not matter.

The rate (39) depends in a global way on the map $f(x)$ through $\rho(x^*)$ and on the noise distribution $P(\xi, x)$ through $h_\infty(x)$. For the rest, only local properties of the map $f(x)$ at 0, x^*, and 1 enter the rate formula. Considered as a function of Δ and σ, the rate (39) has the form of a scaling law.[4, 11] The critical exponents are $1/z$ and 1, the scaling function is $F(x)$, and k, Δ, and σ play the roles of order parameter, control parameter, and relevant scaling field, respectively.[38] It can be shown that (23) is a necessary condition not only for our result (39), but for the validity of *any* scaling law for the rate.

Since (33) and (34) stay valid in the limit $i \to \infty$, it follows that for $x \gg 1$ only y values from a finite neighborhood of $y = x$ notably contribute to the integral in (40). In this domain one can approximate $y^{1/z}$ by $x^{1/z}$. Using (33) once more, one recovers the correct deterministic limit[29] (see also Section 5)

$$k = 2\rho(x^*)(\Delta/b)^{1/z} \qquad \text{for} \quad \Delta \gg \sigma \tag{44}$$

[Since $\Delta > 0$ the σ values are not restricted by (23).] In combination with (40), we thus can conclude that for arbitrary maps and noise distributions the scaling function $F(x)$ is strictly monotonically increasing with x, approaching 0 for $x \to -\infty$ and $2x^{1/z} + o(x^{1/z})$ for $x \to \infty$.

4. EXAMPLES AND NUMERICAL RESULTS

In this section we discuss the central rate formula (39) in more detail for the special case of multiplicative noise (10) of the form (11)–(15). For this purpose it is useful to introduce the quantities

$$T_i := g(f^i_{\Delta=0}(x^*)) \Big/ \frac{d}{dx} f^i_{\Delta=0}(x = x^*) \tag{45}$$

or, equivalently,

$$T_0 = g(x^*), \qquad T_1 = \frac{g(1)}{f'(1)}, \qquad T_i = \frac{g(0)}{f'(1)\, f'(0)^{i-1}} \qquad \text{for} \quad i \geq 2 \tag{46}$$

Whenever a T_i is zero the recursion relation (30) takes the trivial form $h_i(x) = h_{i-1}(x)$. Thus, in order to determine $h_\infty(x)$, such an i value can simply be omitted and all the following indices i have to be reduced by 1. Since the quantities T_i, $i \geq 2$, always vanish simultaneously, the function $h_\infty(x)$ is obtained after a single iteration of the recursion relation (30) in this case. Therefore, we restrict ourselves to the case that $T_i \neq 0$ for all $i \geq 0$ in the following. Note also that $h_m(x)$, $m \geq 0$, can be obtained by formally setting $T_i = 0$ for all $i > m$ in the expressions for $h_\infty(x)$ we will derive in the sequel.

4.1. Lévy Distributions

The most suitable examples for an explicit evaluation of $h_\infty(x)$ in (43) are the symmetric Lévy distributions (11). From the definitions (10) and (42) one sees that $\tilde{P}(q, x) = \exp\{-|g(x)\, q|^\mu\}$ and thus

$$h_\infty(x) = \frac{P_M(x/U_\mu)}{U_\mu} = \int_{-\infty}^{\infty} \frac{dq}{2\pi}\, e^{iqx} e^{-|qU_\mu|^\mu} \tag{47}$$

$$U_\mu := \left(\sum_{i=0}^{\infty} |T_i|^\mu\right)^{1/\mu} = \left(|g(x^*)|^\mu + \left|\frac{g(1)}{f'(1)}\right|^\mu + \frac{|g(0)|^\mu}{|f'(1)|^\mu\, (f'(0)^\mu - 1)}\right)^{1/\mu} \tag{48}$$

Consequently, the rate (39) can be rewritten in the form

$$k = \rho(x^*)\left(\frac{U_\mu}{b}\,\sigma\right)^{1/z} \tilde{F}\left(\frac{1}{U_\mu}\frac{\Delta}{\sigma}\right) \tag{49}$$

$$\tilde{F}(x) := 2\int_0^\infty dy\, y^{1/z} P_M(y - x) \tag{50}$$

We recall that $P_M(\xi)$ in (50) is given by (11) for general μ and alternatively by (12) and (13) in the particular cases of Gaussian ($\mu = 2$) and Lorentz ($\mu = 1$) distributed noise, respectively. Moreover, for $\mu < 2$ the z values are restricted by the condition $z > 1/\mu$ as mentioned below Eq. (13). It is remarkable that in the scaling law (49) *both the critical exponents and the scaling function* $\tilde{F}(x)$ *are universal for any fixed* μ *and* z. As in the theory of critical phenomena,[38] the specific properties of the map $f(x)$ and the coupling function $g(x)$ of the multiplicative noise enter the scaling law (49) only through the nonuniversal scaling amplitudes $\rho(x^*)(U_\mu/b)^{1/z}$ and $1/U_\mu$. In other words, μ and z uniquely determine the universality class and thus play the same roles as the dimensionality of space and the number of components of the order parameter in the theory of critical phenomena.[38] It can be shown that this particular feature is lost as soon one goes beyond the realm of Lévy distributions.

4.2. Dichotomous Noise

For dichotomous white noise (14) one finds that $\tilde{P}(q, x) = \cos(|g(x)|\, q)$. With (43) one arrives at

$$h_\infty(x) = \lim_{m \to \infty} \frac{1}{2^{m+1}} \sum_{\sigma_i = \pm 1} \delta\left(x - \sum_{i=0}^{m} \sigma_i T_i\right) \tag{51}$$

where the outer sum runs over all the 2^{m+1} possible configurations of the "spins" σ_i, $0 \leqslant i \leqslant m$. Note that $h_\infty(x) = 0$ for $x > \sum_{i=0}^{\infty} |T_i|$. Further details of $h_\infty(x)$ crucially depend on the specific values of the T_i. For instance, in the case of additive noise, $g(x) \equiv 1$ and a map with $f'(0) = -f'(1)$ the function $h_\infty(x)$ consists of two δ-peaks if $f'(0) = \infty$, an infinity of δ-peaks located on a Cantor dust if $2 < f'(0) < \infty$, and is equal to $\Theta(|x| - 2)/4$ if $f'(0) = 2$, where $\Theta(x) := \int_{-\infty}^{x} \delta(y)\, dy$ is the Heaviside function. For $1 < f'(0) < 2$ the features of $h_\infty(x)$ are not so obvious and are not discussed here in further detail.

Introducing (51) into (40) yields

$$F(x) = \lim_{m \to \infty} \frac{1}{2^m} \sum_{\sigma_i = \pm 1} \left| x - \sum_{i=0}^{m} \sigma_i T_i \right|^{1/z} \Theta\left(x - \sum_{i=0}^{m} \sigma_i T_i \right) \tag{52}$$

For $x < -\sum_{i=0}^{\infty} |T_i|$ we have $F(x) = 0$, implying for the rate (39) that

$$k = 0 \qquad \text{for} \quad \Delta/\sigma \leqslant -\sum_{i=0}^{\infty} |T_i| \tag{53}$$

Basically, this is a consequence of the fact that the noise distribution (14) has bounded support and thus for $\Delta < 0$ and sufficiently small σ particles cannot escape from $[0, 1]$. For large $x = \Delta/\sigma$ one recovers the deterministic limit (44) from (39) and (52).

4.3. Exponential Noise Distributions

For the exponential noise distributions (15), closed expressions for $h_\infty(x)$ in (43) can be given in the three particular cases of Gaussian ($\alpha = 2$), ordinary exponential ($\alpha = 1$), and confined homogeneous noise ($\alpha = \infty$). The case of Gaussian noise readily follows from (47)–(50) with $\mu = 2$. For $\alpha = 1$ one finds that $\tilde{P}(q, x) = (1 + [g(x)\, q]^2)^{-1}$ and thus by invoking the residue theorem that

$$h_\infty(x) = \sum_{i=0}^{\infty} \frac{e^{-|x/T_i|}}{2\,|T_i|} \prod_{\substack{l=0 \\ l \neq i}}^{\infty} \frac{T_i^2}{T_i^2 - T_l^2} \tag{54}$$

Introducing this expression into (40) does not allow a further evaluation of $F(x)$ except for $x \gg 1$, corresponding to the deterministic limit of the rate (44), and for $x \ll 1$, where one finds that

$$F(x) = \Gamma\left(1 + \frac{1}{z}\right) \sum_{i=0}^{\infty} |T_i|^{1/z}\, e^{-|x/T_i|} \prod_{\substack{l=0 \\ l \neq i}}^{\infty} \frac{T_i^2}{T_i^2 - T_l^2} \tag{55}$$

It can be shown that both in (54) and (55) the infinite sum converges to a finite value for any x, including $x = 0$, and that both results stay valid even when $T_i^2 \to T_l^2$ for some $i \neq l$.

For $\alpha=\infty$ one finds that $\tilde{P}(q,x)=\sin[g(x)\,q]/[g(x)\,q]$. Evaluating (43) by means of the residue theorem then leads to a somewhat similar result as in (51), namely

$$h_\infty(x)=\lim_{m\to\infty}\frac{1}{2^{m+2}}\sum_{\sigma_i=\pm 1}\frac{|x+\sum_{i=0}^m \sigma_i T_i|\,(x+\sum_{i=0}^m \sigma_i T_i)^{m-1}}{m!\prod_{i=0}^m \sigma_i\,|T_i|} \tag{56}$$

In further similarity with (51), $h_\infty(x)=0$ for $x>\sum_{i=0}^\infty |T_i|$, which is not obvious from (56) but readily follows from the recursion relation (30). [In the same way one finds that $h_\infty(x)=1/2$ for $x<1-\sum_{i=1}^\infty |T_i|$ provided $\sum_{i=1}^\infty |T_i|\leqslant 1$.] As for (54), introducing (56) into (40) does not allow a further evaluation of $F(x)$ except for sufficiently large positive and negative x, for which one recovers (44) and (53), respectively.

For the exponential noise distributions (15) with general $\alpha>0$, the rate (39) can be further evaluated only in the deterministic limit $\Delta\gg\sigma$ [see (44)] and in the opposite asymptotic regime $\Delta\ll-\sigma$ [but still respecting the condition (23)]. This case $\Delta\ll-\sigma$ is considered in the remainder of this subsection. By means of a technically involved but somewhat tedious calculation an asymptotically exact analytical solution of the recursion (30), (31) for $h_i(x)$ is possible for large $|x|$, yielding

$$h_\infty(x)=e^{-|x/A_\infty|^\alpha}\frac{\alpha}{2\Gamma(1/\alpha)\,A_\infty} \qquad \text{for}\quad 0<\alpha<1 \tag{57}$$

$$h_\infty(x)=\frac{e^{-|x/A_\infty|}}{2A_\infty}\prod_{\substack{i=0\\ T_i^2\neq A_\infty^2}}^{\infty}\frac{A_\infty^2}{A_\infty^2-T_i^2} \qquad \text{for}\quad \alpha=1 \tag{58}$$

$$h_\infty(x)=e^{-|x/A_\infty|^\alpha}K\,|x|^{\beta\ln|x|+\gamma} \qquad \text{for}\quad \alpha>1 \tag{59}$$

where we introduced

$$A_\infty=\max_i |T_i|=\max\left\{|g(x^*)|,\left|\frac{g(1)}{f'(1)}\right|,\left|\frac{g(0)}{f'(0)\,f'(1)}\right|\right\}$$

$$\text{for}\quad 0<\alpha\leqslant 1 \tag{60}$$

$$A_\infty=\left(\sum_{i=0}^{\infty}|T_i|^{\alpha/(\alpha-1)}\right)^{(\alpha-1)/\alpha}=U_{\mu=\alpha/(\alpha-1)} \qquad \text{for}\quad \alpha>1 \tag{61}$$

$$\beta=(\alpha-1)(2-\alpha)/[4\ln f'(0)] \tag{62}$$

$$\gamma=\frac{\alpha-1}{2\ln f'(0)}\ln\left(\frac{\alpha\pi/\Gamma(1/\alpha)^2}{2(\alpha-1)}\left|\frac{g(0)\sqrt{f'(0)}}{A_\infty^\alpha f'(1)}\right|^{(2-\alpha)/(\alpha-1)}\right) \tag{63}$$

Additionally, the nongeneric cases that $|T_0| = |T_1|$, $|T_0| < |T_1| = |T_2|$, or $|T_0| = |T_2| > |T_1|$ are excluded for the sake of convenience when $\alpha \leqslant 1$. The explicit form of U_μ from (61) is given in (48). The coefficient K in (59) is x-independent, positive, and finite, but cannot be explicitly determined by analytical means except for Gaussian noise: $K = [\sqrt{\pi}\, A_\infty]^{-1}$ for $\alpha = 2$, in agreement with the exact result (47) for $\mu = 2$. Similarly, for $\alpha = 1$ the asymptotic behavior (58), (60) coincides with the exact result (54) for large $|x|$. In the limits $\alpha \to 1$ and $\alpha \to \infty$ the exponentially leading part $\exp\{-|x/A_\infty|^\alpha\}$ in (57) and (59) is in agreement with the exact results (58) and $h_\infty(x) = 0$ for $|x| > \sum_{i=0}^{\infty} |T_i|$ [see the discussion below Eq. (56)], respectively. However, the preexponential contributions in (57) and (59) are no longer valid in these limits, due to the fact that the limit $|x| \to \infty$ does not commute with $\alpha \to 1$ or $\alpha \to \infty$. For similar reasons, the limits $f'(0) \to \infty$ and $f'(1) \to -\infty$ are not admitted when $\alpha > 1$ [in fact, it is only γ in (59) that does not converge to its correct limit 0].

Using these results (57)–(59) for $h_\infty(x)$, one finds for the scaling function $F(x)$ from (40) that

$$F(x) = 2\Gamma\left(1 + \frac{1}{z}\right)\left(\frac{A_\infty^\alpha}{\alpha\,|x|^{\alpha-1}}\right)^{1+1/z} h_\infty(x) \qquad \text{for} \quad x \leqslant -1 \qquad (64)$$

Consequently, for $\Delta \leqslant -\sigma$ *the rate* (39) *is dominated by an Arrhenius-like factor* $\exp\{-|\Delta/(\sigma A_\infty)|^\alpha\}$. This property, but without the explicit expressions for A_∞, was recently derived by Hamm *et al.*[15] It is only for Gaussian noise ($\alpha = 2$) that the exact value of A_∞ was determined previously by Beale[8] [in agreement with our result (61)]. An approximation for general α is given in ref. 39. The preexponential part of the rate (39) becomes proportional to $\sigma^{1/z}|\Delta/\sigma|^{(1-\alpha)(1+1/z)}$ for $\alpha \leqslant 1$ and $\alpha = 2$ according to (57)–(59), (64), i.e., it depends algebraically upon σ and Δ. On the other hand, *for* $\alpha > 1$, $\alpha \neq 2$, *the preexponential dependence of the rate upon* Δ *and* σ *is stronger than any power law according to* (59) *and* (64). Hence, there may be considerable deviations from the exponentially leading Arrhenius law[15] even for rather large negative values of Δ/σ.

4.4. Numerical Simulations

It is beyond the scope of this paper to carry out a systematic numerical verification of the various results derived in this section. For the case of additive Gaussian white and colored noise acting on different kinds of maps $f(x)$ we refer to I and refs. 22, 24. Here, we restrict ourselves to one particular example of non-Gaussian multiplicative noise. For the sake of convenience we consider the logistic map (5) with confined multiplicative

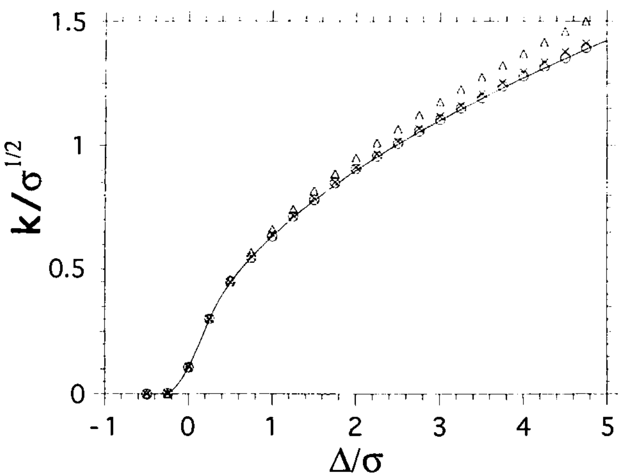

Fig. 1. The escape rate k as a function of the parameter Δ for the logistic map (5) disturbed by multiplicative noise (10) with the noise coupling function $g(x)$ from (65) and a confined homogeneous noise distribution $P_M(\xi) = \Theta(1 - |\xi|)/2$ [corresponding to (15) with $\alpha = \infty$]. The symbols represent the results from numerical simulations of the Langevin equation (1) with noise strengths $\sigma = 10^{-4}$ (triangles), $\sigma = 3.0 \times 10^{-6}$ (crosses), and $\sigma = 10^{-7}$ (circles). The solid line is the theoretical prediction according to (39)–(43) [with z, b, and $\rho(x^*)$ following from (5) and (6)]. Apart from finite σ and Δ effects, the agreement is excellent.

noise according to (10) and (15) with $\alpha = \infty$. As noise coupling function we choose

$$g(x) = \frac{-3(x - 1/2)}{|x - 1/2| + 1} \tag{65}$$

Thus, $g(x)$ is of odd symmetry about the maximum $x^* = 1/2$ of the map $f(x)$, $g(1/2 + x) = -g(1/2 - x)$, and bounded on **R**. Specifically, we have $g(0) = 1$, $g(1/2) = 0$, and $g(1) = -1$. This choice is of particular interest singe it corresponds to a noise distribution (10) that is discontinuous in ξ for any $x \neq x^*$ and exhibits a δ-peak at $\xi = 0$ for $x = x^*$. Thus, the condition (9) is only satisfied in the most general version, namely in the sense of distributions. The theoretical rate is completely fixed by the rate formula (39), (40), the invariant density (6), and the explicit expression (56) for $h_\infty(x)$. The comparison with numerical simulations is shown in Fig. 1.

5. STABILIZATION OF TRANSIENT CHAOS BY NOISE

In this section we consider the escape rate as a function of the noise strength σ for an arbitrary but fixed (small) value of the parameter Δ. To this end we rewrite the rate formula (39), (40) in the form

$$k(\sigma) = 2p(x^*)(|\Delta|/b)^{1/z}\, G(\sigma/\Delta, z) \tag{66}$$

$$G(x, z) := \int_{-1/x}^{\infty} dy\, |1 + xy|^{1/z}\, h_\infty(y) \tag{67}$$

For the sake of simplicity we assume that the order z of the maximum of the map $f(x)$ can be varied without changing the position x^* of this maximum and the values of the slopes $f'(0)$ and $f'(1)$. Consequently, the function $h_\infty(x)$ in (67) is independent of z according to the recursion relation (30), (31). From the normalization (33) of $h_\infty(x)$ it follows that $G(0^+, z) = 1$, in agreement with the deterministic limit (44) for $\Delta > 0$, $\sigma \to 0$. Note that, at variance with I, the factor 2 in (66) has not been absorbed into the definition of $G(x, z)$, since this convention implies the appealing relation

$$k(\sigma)/k(\sigma = 0) = G(\sigma/\Delta, z) \qquad \text{for} \quad \Delta > 0 \tag{68}$$

In agreement with the fact that for $\Delta < 0$ particles cannot escape from the unit interval in the deterministic limit $\sigma \to 0$, we find that $G(0^-, z) = 0$. The discontinuity of $G(x, z)$ at $x = 0$ is of no relevance since the argument $x = \sigma/\Delta$ in (66) never crosses $x = 0$ according to the restrictions (2), (4), and (23) on σ and Δ.

From (67) we can infer that

$$\lim_{x \to \infty} G(x, z) \geqslant \lim_{x \to \infty} x^{1/z} \int_0^{\infty} dy\, y^{1/z} h_\infty(y) = \infty \tag{69}$$

where in the last equality we disregarded the rather strange case that $h_\infty(y) \equiv 0$ for $y \geqslant 0$, which would imply that the escape rate (39), (40) vanishes at the crisis Δ even for finite σ. Further, it is possible to show by closer inspection of (67) that

$$\lim_{z \to \infty} G(x = cz, z) = \int_0^{\infty} dy\, h_\infty(y) \tag{70}$$

for any $c > 0$. Disregarding once more a rather strange case, namely $h_\infty(y) \equiv 0$ for $y < 0$, one sees from (67) and (70) that $G(cz, z) < G(0^+, z) = 1$ for sufficiently large z. With (69) it then follows that $G(x, z)$ has an absolute minimum at $x_{\min}(z) > 0$ in the region $0 \leqslant x < \infty$ for sufficiently large z, say $z > z_0$. In view of (68) this leads to the remarkable conclusion that *transient chaos* ($\Delta > 0$) *is stabilized by noise of appropriate strength* σ *for* $z > z_0$. In this reasoning it is of course tacitly understood that Δ is sufficiently small in order that (68) is still valid for the large values of

$x = \sigma/\Delta$ appearing in (69) and (70). Within this restriction, the maximal reduction of the rate

$$\frac{\min_\sigma k(\sigma)}{k(\sigma=0)} = \frac{k(x_{\min}(z)\,\Delta)}{k(\sigma=0)} = G(x_{\min}(z), z), \qquad z > z_0 \tag{71}$$

is independent of Δ. This amazing effect was observed for the first time using confined homogeneous noise (15) with $\alpha = \infty$ by Franaszek[10] and was predicted analytically and confirmed numerically for Gaussian white and colored noise in I. For related phenomena occurring in the context of deterministic diffusion with noise and two-dimensional noisy maps see refs. 22 and 5, respectively. The present investigation shows that *stabilization of transient chaos can be induced basically by any kind of uncorrelated weak noise*. In particular, the distribution of the noise $P(\xi, x)$ may be asymmetric about $\xi = 0$.

By similar calculations, further qualitative features of $G(x, z)$ can be determined. For convenience we restrict ourselves to the case that $h_\infty(x)$ is bounded (e.g., δ-peaks are thus excluded) and we omit detailed proofs. For particular examples we refer to Fig. 4 in I. One finds that $G(x, z)$ is strictly monotonically decreasing with increasing x in the region $x < 0$ and we thus concentrate on $x \geqslant 0$. For $z \geqslant z_0$ the absolute minimum (in the domain $x \geqslant 0$) of $G(x, z)$ at $x = x_{\min}(z) > 0$ turns out to increase monotonically with z, while $G(x_{\min}(z), z)$ decreases monotonically with z. In particular, one can assume that z_0 is chosen such that the absolute minimum of $G(x, z)$ as a function of $x \geqslant 0$ is at $x = 0$ for all $z < z_0$ and at $0 < x_{\min}(z) < \infty$ for all $z > z_0$. Further, one finds for asymptotically large z that

$$x_{\min}(z) = \frac{z h_\infty(0)}{\int_0^\infty dy\, h_\infty(y)\, y^{1/z}} \tag{72}$$

$$G(x_{\min}(z), z) = \int_0^\infty dy\, h_\infty(y) \tag{73}$$

If $h_\infty(x)$ is symmetric about $x = 0$, it follows from (71) and (73) that

$$\min_\sigma k(\sigma) \to k(\sigma=0)/2 \qquad \text{for large} \quad z \tag{74}$$

Recall that $G(x_{\min}(z), z)$ is strictly decreasing with z and that symmetric noise distributions $P(-\xi, x) = P(\xi, x)$, for instance (11)–(15), lead to a symmetric $h_\infty(x)$. Therefore, *for symmetric noise distributions $P(-\xi, x) = P(\xi, x)$ the stabilization of transient chaos by noise* (71) *cannot exceed a factor of* 1/2 *and this limit is actually saturated when $z \to \infty$ independently* *of any further details of the map* $f(x)$ *and the noise* $P(\xi, x)$. Finally, under the assumption that $\langle y \rangle := \int_{-\infty}^{\infty} y h_\infty(y)\, dy$ is finite, one can show that $z_0 > 1$ if $\langle y \rangle > 0$, $z_0 = 1$ if $\langle y \rangle = 0$, and $z_0 < 1$ if $\langle y \rangle < 0$. In particular, *for symmetric noise distributions, noise-induced stabilization of transient chaos occurs if and only if* $z > 1$. If $\langle y \rangle$ does not exist, things become more complicated.

5.1. Heuristic Explanation

In order to explain noise-induced stabilization of transient chaos, we first consider the case of transient chaos $\Delta > 0$ without noise $\sigma = 0$. Then, a particle (1) escapes from the unit interval $[0, 1]$ if and only if it passes through the small neighborhood of the maximum x^* which is mapped outside $[0, 1]$ under $f(x)$. According to (3), the size of this neighborhood is $L = 2(\Delta/b)^{1/z}$. As we have seen in Eq. (22), the quasi-invariant density is approximated very well by the constant value $\rho(x^*)$ within this entire neighborhood of x^*. Consequently, the probability per time step to escape from the unit interval, i.e., the escape rate k, is given by $k = \rho(x^*)\, L$, in agreement with (44). Next we disturb the map $f(x)$ by a small amount of additive dichotomous white noise (10), (14) with $g(x) \equiv 1$. Thus, at any time step the dynamics is governed with probability 1/2 by either of the maps $f(x) + \sigma$ or $f(x) - \sigma$. Therefore, the size of the respective neighborhoods of x^* which must be visited by an escaping particle is $L_+ = 2[(\Delta + \sigma)/b]^{1/z}$ or $L_- = 2[(\Delta - \sigma)\, \Theta(\Delta - \sigma)/b]^{1/z}$ (we remember that only $\Delta > 0$ is considered). Again, within these neighborhoods of x^*, the probability density is well approximated by $\rho(x^*)$ for small σ and Δ and thus the escape rate is given by

$$k(\sigma) = \rho(x^*)\, \frac{L_+ + L_-}{2} = \rho(x^*)\, \frac{[\Delta + \sigma]^{1/z} + [(\Delta - \sigma)\, \Theta(\Delta - \sigma)]^{1/z}}{b^{1/z}} \tag{75}$$

For $\sigma = 0$ once recovers (44). When σ increases, L_+ grows and L_- shrinks, but *since the* σ *dependence is nonlinear, the decrease of* L_- *exceeds the increase of* L_+ *for* $z > 1$ and sufficiently small σ. This is the basic mechanism leading to a stabilization of transient chaos by noise. It is suggestive that qualitatively the same behavior will be observed for more general noise distributions $P(\xi, x)$.

However, it must be emphasized[11, 12] that the arguments leading to the rate formula (75) are strictly valid only if either $f'(x)$ diverges or $f(x)$ is discontinuous at the interval boundaries $x = 0$ and $x = 1$. Otherwise, the dichotomous noise can drive particles out of the unit interval even if they never visited the appropriate neighborhoods L_+ or L_- of x^*. On the other hand, particles which already left the unit interval can be pushed back into [0, 1] by the noise. A careful inspection of these two competing effects sooner or later leads back to the rather involved calculations of Section 3. It is not even possible to decide by simple arguments whether the two effects sum up to an effective enhancement or reduction of the naive result (75).

The line of reasoning yielding (75) for dichotomous white noise can readily be generalized to arbitrary noise distributions $P(\xi, x)$ with the result

$$k(\sigma) = \rho(x^*) \int_{-\infty}^{\infty} 2\left(\frac{\Delta + \sigma\xi}{b}\right)^{1/z} \Theta(\Delta + \sigma\xi)\, P(\xi, x^*)\, d\xi \tag{76}$$

[Unlike in (75), negative Δ are now admitted.] Similar rate formulas have been derived in refs. 3, 4, and 11. Approximations similar in spirit have also been used in refs. 9 and 21. Equation (76) can be recast into the form (39), (40), but with $P(x, x^*)$ in place of $h_\infty(x)$. According to (30), (31), the naive rate (76) is thus equivalent to approximating $h_\infty(x)$ by $h_0(x)$ and becomes exact only if $P(\xi, x) = \delta(\xi)$ or $f'(x)^{-1} = 0$ at the interval boundaries $x = 0$ *and* $x = 1$. For special maps and noises, improved approximations, comparable to approximating $h_\infty(x)$ by $h_1(x)$ in our notation have been obtained in refs. 3 and 11. One can easily find both examples for which these approximations of $h_\infty(x)$ by $h_0(x)$ or $h_1(x)$ agree very well or very badly with the exact expression (39). Several of the above-mentioned references are also concerned with problems other than the escape rate for a noisy one-dimensional map near a crisis. Their merits in this respect are of course untouched by our discussion.

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